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# Dynamic Modeling of Raw Material Taxation in the Transition to a Circular Economy: An Optimal Control Framework for Toxic Externality Allocation

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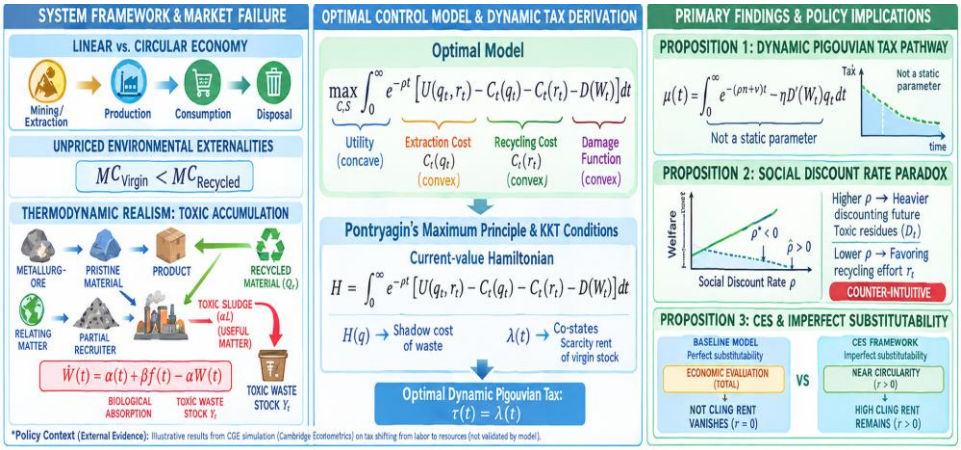
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*A B S T R A C T*

This paper presents a continuous-time optimal control model to determine the dynamic extraction taxes required to transition from a linear to a circular economy. It addresses the market failure where unpriced environmental externalities make virgin materials artificially cheaper than recycled alternatives.The analysis makes four main arguments: Dynamic Taxation: Using Pontryagin's Maximum Principle, the model demonstrates that optimal Pigouvian taxes on virgin materials must match the discounted present value of all future marginal environmental damages. Static taxes are inadequate. 1. Discount Rate Paradox: Counterintuitively, higher social discount rates increase optimal recycling intensity. This occurs because present-biased planners heavily discount the future environmental costs associated with toxic recycling residues. 2. Imperfect Substitution: Applying a CES production framework reveals that if virgin and recycled inputs are imperfect substitutes, 100% circularity is impossible. A strictly positive rate of virgin extraction must always remain part of the optimal path.3. Thermodynamic Realism: Instead of generic waste stocks, the model tracks specific toxic pollutant accumulation, capturing both primary depletion and the contamination inherent in secondary processing. Theoretical findings suggest that environmental tax reform specifically shifting the fiscal burden from labor to natural resource extraction can plausibly support ecological sustainability without triggering macroeconomic contraction. However, achieving this requires dynamically adjusted fiscal instruments that fully internalize the intertemporal shadow costs of environmental degradation.




## 1. Introduction

Contemporary industrial economies remain structurally anchored to a linear model of extraction, production, consumption, and disposal [1]. Although this configuration has driven extraordinary capital accumulation and technological growth over the past two centuries, it has concurrently imposed compounding, potentially irreversible ecological costs on the biosphere. A central structural anomaly within this system is that virgin raw materials consistently outcompete recycled alternatives on price not because of any intrinsic material superiority, but because primary extraction has historically benefited from entrenched infrastructural economies of scale and, more critically, from an implicit environmental subsidy: firms freely externalize the costs of both extraction damage and terminal waste disposal onto the natural environment [2]. Secondary recycling, by contrast, operates against persistent structural disadvantages fragmented reverse logistics, feedstock contamination, polymer degradation, and high sorting costs that keep the marginal cost of high-purity recycled inputs above that of virgin equivalents. The result is a stable market equilibrium that systematically rewards depletion and penalizes

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circularity.Environmental economists and policymakers have proposed the upstream raw material tax variously termed a virgin material tax or primary resource levy as a direct fiscal instrument to correct this price distortion, which raises the relative price of virgin inputs and stimulates substitution toward secondary materials [3]. The authors argue, however, that existing analytical treatments of this instrument remain inadequate. Despite growing international legislative interest, most notably articulated in the European Union's Circular Economy Action Plan and the United Nations' Agenda 2030 [4], static, single-period Pigouvian approximations cannot capture the temporal complexity introduced by the continuous depletion of non-renewable geological stocks and the intergenerational accumulation of toxic pollutants. Both dynamics require the tax to be conceived as a continuously time-varying trajectory rather than a fixed levy. This paper addresses that analytical gap by developing a continuous-time optimal control framework that models the macroeconomy as an integrated ecological-economic system. Its primary objective is to derive the explicit, infinite-horizon trajectory of the optimal upstream raw material tax. A secondary objective is to determine how chronic toxic waste accumulation modifies or displaces the classical Hotelling rule governing non-renewable resource scarcity. The framework ultimately aims to identify the precise dynamic tax path capable of guiding the economy along a saddle-path stable transition toward a fully circular macroeconomic equilibrium.

The development of a rigorous continuous-time optimal taxation model for circular economy transitions necessitates comprehensive grounding in natural resource economics, endogenous growth theory, and environmental fiscal policy. This literature review synthesizes the theoretical foundations and empirical constraints governing dynamic resource taxation.

### *1.1. Classical Scarcity and the Hotelling Framework*

The canonical foundation of exhaustible resource economics originates with Hotelling's seminal rule, which establishes that under perfect competition and frictionless markets, the scarcity rent of a finite geological stock must appreciate exponentially at the prevailing social discount rate [5]. This result formalizes the intertemporal arbitrage condition: rational resource owners will defer extraction only if the expected capital gain from future price appreciation equals or exceeds alternative financial returns. However, Hotelling's original framework operates in an environmental vacuum, abstracting entirely from extraction externalities, processing emissions, and post-consumption waste accumulation. In contemporary macroeconomic contexts where primary inputs face substitution from secondary recycling and where toxic waste imposes severe intergenerational costs, the classical Hotelling paradigm requires fundamental extension. Modern resource economics must transition from managing input scarcity to managing the biosphere's finite assimilative capacity for industrial waste a shift that reorients optimal control modeling from geological depletion constraints to ecological absorption constraints.

### *1.2. Optimal Growth, Waste Accumulation, and Recycling Dynamics*

The incorporation of polluting waste as an explicit dynamic state variable in infinite-horizon optimization has emerged as a critical modeling paradigm. and Boucekkine and El Ouardighi introduced waste accumulation into an endogenous AK growth framework, revealing a counter-intuitive paradox: when recycling revenues are fully channeled into capital accumulation, the economy experiences an unsustainable boom followed by long-run contraction in both consumption and capital stock. This finding challenges the prevailing assumption that recycling invariably supports perpetual growth. Their framework further modeled the environmental absorption capacity for persistent pollutants such as synthetic polymers as effectively zero, reflecting the near-infinite biological degradation timeframes and forcing active, costly intervention rather than passive natural abatement [2].

Subsequent research by Lafforgue and Lorang extended this analysis to incorporate simultaneous environmental, climate, and resource constraints [6, 7]. Employing Herfindahl-style optimization to characterize least-cost material substitution, they demonstrated that recycling is not uniformly clean. This finding underscores the necessity of precisely calibrated tax-subsidy instruments to manage non-monotonic transitions.

Bertarelli, Lodi, and Ragni applied finite-horizon optimal control to model labor allocation between primary and secondary sectors, emphasizing the dual nature of waste as both environmental liability and recyclable feedstock endowment. Using a linear production structure consistent with UN Agenda 2030 transition timelines, they introduced a critical parameter $\boldsymbol{\varphi}$ governing whether waste's negative externality dominates its positive resource endowment effect, thereby determining optimal intersectoral labor allocation [4].

### *1.3. Circular Economy Taxation and Fiscal Policy Instruments*

Empirical analysis of upstream resource taxation reveals significant design challenges. Söderholm's evaluation of aggregate extraction taxes on gravel and rock across the UK, Sweden, and Denmark demonstrated that while upstream virgin material taxes reduce primary extraction, they function as second-best instruments when deployed in isolation. The transmission mechanism is structurally weak: upstream price signals provide insufficient incentive for decentralized household-level waste sorting or municipal recycling investment. Without complementary downstream instruments such as Advance Disposal Fees or terminal Disposal Fees the circular loop remains incomplete [1].

Milios addressed this governance gap by proposing a comprehensive, multi-tiered taxation framework spanning the entire product lifecycle: a natural resource tax at primary extraction, VAT relief for reuse and repair services during product use, and a tiered "waste hierarchy tax" at end-of-life [3]. Simulations calibrated to Swedish waste infrastructure illustrate that differential taxation across disposal alternatives penalizing landfilling heavily (e.g., 500 SEK/tonne), taxing incineration moderately (e.g., 300 SEK/tonne), and subsidizing recycling (e.g., 100 SEK/tonne or less) is essential for closing the material loop. Milios further argues that environmental taxes are "growth-friendly" relative to labor and income taxes, generating lower deadweight efficiency losses [3].

Table 1 provides a structurally synthesized overview of the foundational academic literature positioning the theoretical model developed in the subsequent sections of this report.

**Table 1.** Synthesis of key theoretical and empirical literature shaping the continuous-time optimal control framework. Data compiled from respective sources.

| Author(s) & Year | Primary Focus & Methodological Approach | Key Finding / Economic Insight |
|---|---|---|
| Hotelling [5] | Resource scarcity; dynamic equilibrium pricing | Scarcity rent must appreciate exponentially at the prevailing social discount rate. |
| Boucekkine & El Ouardighi [2] | Optimal endogenous growth; zero-absorption waste | Recycling operating strictly as an income generator devoted to capital accumulation can trigger long-run macroeconomic contraction. |
| Söderholm [1] | Empirical taxation on virgin aggregates in Europe | Upstream taxes alone are second-best; complementary downstream incentives (ADF/DF) are required. |
| Milios [3] | Comprehensive Circular Economy Taxation Framework | Tiered waste hierarchy taxes across three lifecycle stages are necessary to penalize terminal disposal; environmental taxes are growth-friendly. |
| Lafforgue & Lorang[7] | Dynamic recycling modeling with severe climate constraints | Recycling follows an inverted U-shaped path due to the inherent GHG emissions of the secondary recovery process. |
| Bertarelli, Lodi, & Ragni [4] | Optimal labor allocation; finite horizons (Agenda 2030) | Waste serves paradoxical dual roles as an environmental hazard and a raw material endowment for secondary sectors. |

Source: Research Findings

### *1.4. Theoretical Advancements of the Proposed Framework*

While the application of optimal control theory to environmental taxation and resource management is well-established in the cited literature, the present analytical manuscript introduces several distinct, targeted theoretical and methodological advancements. These advancements explicitly differentiate the proposed framework from the established models of Boucekkine and El Ouardighi [8] , Lafforgue and Lorang [7] , and Bertarelli et al. [4].

First, rather than modeling recycling as a generalized process that either generates aggregate capital income [2] or inherently reduces gross physical mass within a strictly finite horizon [4], this paper models recycling explicitly through the lens of inescapable thermodynamic entropy evaluated over an infinite horizon. The proposed model isolates a specific parameter, β, which strictly represents the unrecoverable toxic sludge and hazardous pollution generated entirely by the technological and thermodynamic inefficiencies of the secondary recycling process itself. By actively abstracting away from gross physical mass accounting and defining the core state variable entirely around the accumulation of toxic externalities, the model provides a targeted, mathematically rigorous derivation of the environmental penalty associated specifically with secondary material flows.

Second, the present analysis fundamentally extends classical dynamic Pigouvian taxation theory by formally establishing the precise mathematical transversality conditions under which the classical Hotelling scarcity rent vanishes completely. While prior academic literature broadly acknowledges that recycling extends the viable, economically useful life of exhaustible resource endowments, this analysis formally defines and derives the boundary state of “economic exhaustion.” This is a unique theoretical condition where the implementation of the optimal dynamic tax forces the permanent cessation of virgin extraction long before the total physical depletion of the resource in the earth’s crust, thereby permanently overriding the exponential growth of the scarcity rent posited by Hotelling [5].

Third, the analysis provides a rigorous, proposition-based mathematical proof of a counter-intuitive comparative static regarding the intergenerational social discount rate. While standard environmental economics and sustainability science broadly posit that lower discount rates strictly favor ecosystem conservation and higher optimal recycling, this model exactly specifies the mathematical boundary conditions under which a higher intergenerational discount rate actually structurally increases the socially optimal equilibrium recycling effort. This uniquely occurs because the future toxic penalties of the

recycling process itself specifically the delayed environmental impact of the parameter β are heavily discounted by a planner who prioritizes present consumption over future environmental integrity.

These theoretical results should be read against their maintained assumptions rather than as general properties of resource-recycling economies. In particular, the baseline result that virgin extraction can permanently reach zero (Section 5.4) follows directly from the linear aggregator $C(t) = q(t) + r(t)$, which makes virgin and recycled material perfect substitutes; under this assumption a corner solution in $q$ is the expected outcome once the marginal social cost of extraction exceeds that of recycling, and is consistent with the standard optimal-substitution literature on exhaustible resources and backstop technologies. It corresponds to the perfect-substitutes edge of the more general constant-elasticity specification that closely related optimal-control models of recycling deliberately exclude Bertarelli et al. restrict attention to imperfect substitution for exactly this reason rather than being a distinctive feature of the present model. The specific contribution of this paper is, first, to characterize that boundary formally within an infinite-horizon optimal control setting that couples resource depletion with toxic-waste accumulation (Sections 5.1–5.4); second, to show in Section 6 that the corner solution is not robust once virgin and recycled inputs are imperfect substitutes which is where this paper's substantive qualification of the standard result lies, and which parallels the non-vanishing-extraction result of Bertarelli et al. [4] in a different but related environment; and third, to derive in Section 5.5 the market instruments an upstream extraction tax and a downstream recycling-residue tax that decentralize the planner's solution. Claims below are stated with respect to this specific mechanism, not as claims of priority over the broader dynamic-taxation and exhaustible-resource literatures reviewed in Section 2.

## 2. Method

### *2.1. Methodology*

#### *2.1.1. Macroeconomic System Architecture and Optimization Setup*

##### *2.1.1.1.The Central Planner's Objective Function*

To mathematically determine the exact optimal dynamic path of the raw material tax, the macroeconomy is formally modeled as an integrated ecological-economic system directed by a representative social planner. The planner's objective is to discover the continuous control vectors that continuously maximize intergenerational social welfare evaluated over an infinite time horizon. We define social welfare as the discounted value of utility from consumption, minus extraction and logistics costs, mathematically subtracting the physical extraction costs of virgin materials, subtracting the operational logistics costs of secondary recycling, and subtracting the long-term ecological damages resulting from the accumulation of unrecoverable toxic waste.

The continuous-time, infinite-horizon maximization problem is formally presented via the integral equation.

$$\max_{q,r} \int_0^{\infty} e^{-\rho t} [U(q + r) - C_q(q) - C_r(r) - D(W)]dt$$

The structural system variables and their strictly defined functional forms are mathematically established as follows:

- ρ>0 represents the strictly positive intergenerational social discount rate. This parameter reflects society's prevailing financial and psychological time preference for present consumption over future environmental and economic welfare.
- q(t)≥0 serves as the first continuous control variable, representing the instantaneous, non-negative rate of primary virgin raw material extraction.
- r(t)≥0 serves as the second continuous control variable, representing the instantaneous, non-negative secondary recycling effort deployed across the macroeconomy.
- C(t) represents aggregate societal material consumption available for final utility generation.
- U(C) is the societal utility function. We assume the utility function is smooth and concave, such that U' (C)>0 and U″ (C)<0. This mathematically satisfies the standard, fundamental microeconomic assumption of diminishing marginal utility of aggregate consumption.
- Cq(q) represents the aggregate cost function for virgin raw material extraction. We assume the cost function rises at an increasing rate, where $C_{q'}(q)>0$ and $C_{q''}(q)>0$. This explicit convexity dynamically captures the exponentially escalating marginal operational costs associated with extracting increasingly inaccessible, geologically deeper, or lower-grade metallic and mineral deposits.
- $C_r(r)$ represents the operational, energetic, and logistics cost function for the secondary recycling sector. It is also assumed strictly convex, where $C_{r'}(r) > 0$ and $C_{r''}(r) > 0$. This directly reflects the harsh realities of reverse logistics: harvesting concentrated, easily sortable homogenous waste (the "low-hanging fruit" of industrial scrap) is extremely cheap, but mathematically scaling secondary material extraction from deeply embedded, heavily cross-contaminated municipal solid waste requires exponentially higher energy, chemical, and labor inputs.
- D(W) is the environmental damage function associated directly with the physical stock of unrecoverable toxic waste, assumed to be strictly convex, where D'(W)>0 and D″(W)>0. This strict convexity accurately captures the compounding, non-linear degradation of localized and global ecosystems as pollutants systematically accumulate past critical, irreversible biological tipping points.

##### *2.1.1.2 State Variables and Toxic Externality Mechanics*

The physical, objective reality of the defined macroeconomic system is constrained by two defined state variables and their respective continuous differential equations of motion tracking mass and toxicity over time.

The first state variable, S(t), represents the total physical stock of non-renewable virgin raw materials remaining geologically intact in the earth's crust. Assuming fixed geological planetary boundaries, the depletion of this strictly finite resource is governed entirely by the instantaneous extraction rate q(t). The first continuous differential state equation of motion is fundamentally defined as:

$$\dot{S}(t) = -q(t)$$

The second state variable, W(t), represents the aggregate planetary stock of unrecoverable, toxic environmental pollutants. In this precisely targeted externalities accounting framework, virgin extraction q(t) introduces pristine, uncontaminated materials into the economy. Due to the linear nature of primary consumption, these materials ultimately convert into fully polluting elements at the strict end of their linear lifecycle. Conversely, the secondary recycling process r(t) extracts and actively diverts mass away from the general municipal discard pool, effectively intercepting the material before it becomes terminal, landscape-degrading waste.

H owever, the intensive chemical, thermal, and mechanical processing requirements of heavy industrial recycling unavoidably generate a severe toxic sludge byproduct. This toxic residue is strictly and mathematically proportional to the technological inefficiency parameter β∈(0,1), which explicitly represents the inescapable thermodynamic entropy inherent in any industrial process bound by the second law of thermodynamics. The remaining fraction of processed mass, defined mathematically as (1-β)r(t), constitutes inert, useful matter that successfully circulates back within the macroeconomy without imposing new environmental shadow costs.

The continuous differential equation for toxic waste accumulation W(t) must dynamically account for the inflow of primary virgin poll,utants, the inflow of secondary recycling toxic residues (βr(t)), and the natural biological absorption capacity of the global ecosystem. Assuming the ecosystem naturally decays the toxic stock at a constant, proportional biological rate α≥0, the formal differential state equation is:

$$\dot{W}(t) = q(t) + \beta r(t) - \alpha W(t)$$

This explicit, robust formulation clearly and mathematically delineates that while the recycling effort r(t) acts as a substitute for virgin extraction q(t), it is definitively not a perfectly clean "green" technology. It inherently carries its own severe environmental footprint mathematically defined and bounded by βr(t).

##### *2.1.1.3. Baseline Assumptions and Substitutability Constraints*

The baseline model assumes that total aggregate societal consumption is generated by the simple linear sum of virgin extraction flows and recycling flows. This implicitly dictates perfect substitutability between primary and secondary materials within the societal utility function:

$$C(t) = q(t) + r(t)$$

This specific functional assumption dictates that downstream industrial producers and terminal consumers are entirely indifferent between utilizing a newly extracted unit of virgin material and a processed unit of recycled material. While perfect substitutability provides the necessary mathematical tractability for smoothly tracking mass-to-utility conversions and establishing a perfectly circular, mathematically balanced steady state, it remains a robust and somewhat idealized abstraction. In real-world, complex industrial sectors, recycled materials frequently differ substantially from virgin inputs regarding tensile strength, molecular durability, heavy metal contamination risks, and stringent regulatory acceptability (especially prevalent in food-grade packaging or biomedical applications). The intrinsic limitations of this specific assumption, and the complex mathematical consequences of relaxing it via a Constant Elasticity of Substitution (CES) aggregator, will be thoroughly addressed in Section 6 to ensure the utmost theoretical robustness of the primary analytical findings.

## 3. Results and Discussion

### *3.1. Analytical Derivations and Optimal Policy Trajectories*

To appropriately manage the mathematical boundary solutions mapping the macroeconomic transition to a sustainable circular economy specifically

isolating the exact point in time where primary virgin extraction permanently ceases (q=0) we solve the optimization problem using the Pontryagin Maximum Principle and a current-value Hamiltonian incorporating all necessary inequality constraints via Karush-Kuhn-Tucker (KKT) multipliers to handle boundary corner solutions.

Let $\lambda(t)\geq 0$ represent the continuous co-state variable (shadow price) of the exhaustible geological resource stock S(t), and let $\mu(t)\geq 0$ represent the strictly positive shadow cost of the accumulating toxic waste stock W(t). The augmented current-value Hamiltonian H, which represents the instantaneous net benefit to society inclusive of shadow values, is structurally defined as follows:

$$\mathcal{H} = U(q+r) - C_q(q) - C_r(r) - D(W) - \lambda q - \mu(q + \beta r - \alpha W) + \eta_q q + \eta_r r$$

Where $\eta_q(t)$ and $\eta_r(t)$ represent the strict, non-negative KKT multipliers explicitly associated with the control boundary constraints $q(t) \geq 0$ and $r(t) \geq 0$, respectively. To satisfy optimization theory, these terms are strictly subject to the complementary slackness conditions where $\eta_q q = 0$ and $\eta_r r = 0$.

### *3.2. Hamiltonian Formulation and First-Order Conditions*

Taking the partial derivatives of the augmented current-value Hamiltonian with respect to the two continuous control variables yields the necessary first-order conditions (FOCs) required for the socially optimal intertemporal allocation of resources.

Evaluating for the virgin raw material extraction flow q(t):

$$\frac{\partial \mathcal{H}}{\partial q} = U'(q+r) - C_{q'}(q) - \lambda - \mu + \eta_q = 0$$

Evaluating for the secondary recycling effort r(t):

$$\frac{\partial \mathcal{H}}{\partial r} = U'(q+r) - C_{r'}(r) - \beta\mu + \eta_r = 0$$

To fully understand the core, underlying economic dynamics, the analysis first evaluates a standard interior solution scenario where both the virgin mining sector and the circular secondary recycling sector operate simultaneously to satisfy societal demand ($q > 0, r > 0$). Under this specific mathematical condition, the non-negativity boundary constraints do not bind, explicitly meaning the KKT multipliers permanently vanish from the system ($\eta_q = 0, \eta_r = 0$). By substituting and equating the marginal utility terms ($U'(q+r)$) from both first-order conditions, the mathematical derivation yields the system's inter-sectoral arbitrage condition:

$$C_{r'}(r) + \beta\mu = C_{q'}(q) + \lambda + \mu$$

Rearranging the terms to algebraically isolate the marginal operational cost of recycling yields:

$$C_{r'}(r) = C_{q'}(q) + \lambda + (1-\beta)\mu$$

This fundamental algebraic relationship serves as the rule governing materials substitution within the optimal macroeconomy. It strictly dictates that the technological and economic frontier between the primary mining and secondary recycling industries is optimally established exactly when the marginal operational cost of recycling ($C_{r'}$) is perfectly balanced by the total marginal social cost of utilizing virgin raw materials. This total marginal social cost is a complex aggregate encompassing the direct physical extraction costs ($C_{q'}$), the intergenerational opportunity cost of resource depletion representing the Hotelling scarcity rent ($\lambda$), and the net environmental mathematical savings achieved by purposefully diverting material from a fully polluting virgin source to a partially polluting secondary channel ($(1-\beta)\mu$).

### *3.3. Proposition 1: Derivation of the Dynamic Pigouvian Tax*

**Proposition 1.** Under the rigorously defined baseline assumptions of cost function convexity and positive natural biological absorption ($\alpha > 0$), the optimal raw material extraction tax $\tau(t)$ levied upstream must operate as a continuous dynamic pathway equal to the discounted present value of all future marginal environmental damages.

Proof. The explicit dynamic evolution of the environmental shadow price over time is mathematically determined by differentiating the current-value Hamiltonian with respect to the specific state variable for waste. We derive the shadow-cost equation for toxic waste using the standard continuous-time co-state:

$$\dot{\mu}(t) = \rho\mu(t) - \frac{\partial \mathcal{H}}{\partial W}$$

Because the shadow cost $\mu$ is strictly defined as the negative of the standard Hamiltonian shadow value $p_W$ (where $\mu = -p_W$), the partial derivative of the Hamiltonian specifically with respect to the accumulated waste stock W is:

$$\frac{\partial \mathcal{H}}{\partial W} = -D'(W) + \alpha\mu$$

Substituting this derived partial derivative back directly into the primary co-state equation yields:

$$\dot{\mu}(t) = \rho\mu(t) - (-D'(W) + \alpha\mu(t))$$

$$\dot{\mu}(t) = (\rho + \alpha)\mu(t) - D'(W(t))$$

This forms a fundamental linear, first-order, non-homogeneous ordinary differential equation. Solving this complex equation strictly forward in time, subject to the standard transversality boundary condition $\lim_{t\to\infty} e^{-\rho t}\mu(t) = 0$ designed to explicitly prevent explosive, economically nonsensical shadow prices, yields the following definite integral solution:

$$\mu(t) = \int_t^{\infty} e^{-(\rho+\alpha)(s-t)} D'(W(s)) ds$$

This mathematical solution shows that, within the model, the instantaneous shadow cost $\mu(t)$ equals to the total present value of all future marginal damages $D'(W)$ generated over the infinite horizon, rigorously discounted at the combined, effective rate of $(\rho + \alpha)$. This effective discount rate combines the societal intergenerational time preference ($\rho$) with the natural ecosystem's biological absorption capacity ($\alpha$).

To compel decentralized profit-maximizing extraction firms to respect the socially optimal extraction rate strictly defined by the central planner, the planner must fully internalize this shadow cost into market prices. Based entirely on the interior first-order condition for virgin extraction ($U' = C_{q'} + \lambda + \mu$), the optimal Pigouvian tax $\tau(t)$ levied on every newly extracted unit of virgin mass must continuously and equal the derived shadow cost of pollution.

$$\tau(t) = \mu(t)$$

This derivation indicates that optimal environmental taxation is not a static legislative parameter debated in parliaments, but a dynamic mathematical curve intrinsically and continually linked to the ecosystem's evolving biological capacity to safely digest toxic waste.

This equivalence between the optimal tax and the shadow cost of waste is a property of the baseline model rather than a general principle of resource taxation. It relies on two maintained assumptions: that each unit of virgin extraction contributes exactly one unit to the waste stock (Section 4.2), and that the scarcity rent $\lambda(t)$ is separately internalized by extracting firms through their own optimization over the resource stock, so that the tax $\tau(t)$ need only correct the environmental externality this is made explicit in the decentralization result of Section 5.5. Under a different waste-generation technology, or without well-defined property rights over $S$, the extraction tax required to implement the social optimum would generally differ from $\mu(t)$ alone.

### *3.4. Co-State Equation for Resource Scarcity*

The dynamic, intertemporal evolution of the shadow price for the unextracted, primary virgin resource stock is similarly mathematically determined by differentiating the Hamiltonian with respect to the state variable S(t):

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial \mathcal{H}}{\partial S}$$

Because physical extraction costs in this specific formulation of the baseline model are assumed to be independent of the remaining total stock size (meaning $\partial C_q / \partial S = 0$), the partial derivative $\partial \mathcal{H} / \partial S$ mathematically equals zero. Therefore, the complex co-state differential equation simplifies significantly to:

$$\dot{\lambda}(t) = \rho\lambda(t)$$

Integrating this simple, first-order differential equation yields the standard, widely recognized classical Hotelling rule:

$$\lambda(t) = \lambda_0 e^{\rho t}$$

This fundamentally indicates that, barring any extreme boundary conditions, the intrinsic, market-driven scarcity value of the non-renewable resource naturally grows exponentially at a rate precisely matching the prevailing social discount rate, consistent with early resource economics paradigms [5].

### *3.5. Transversality Conditions and Economic Exhaustion*

To guarantee the long-term mathematical viability and stability of the continuous dynamic system and fully evaluate the terminal transition to a perfectly circular economy, the specific transversality boundary conditions evaluated at infinity must be firmly established. The transversality condition required for any non-renewable resource stock over an infinite optimization horizon mathematically dictates that the discounted present value of the remaining physical stock must approach zero as time actively approaches infinity:

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)S(t) = 0$$

Substituting the derived Hotelling exponential trajectory $\lambda(t) = \lambda_0 e^{\rho t}$ directly into the transversality limit yields the following relation:

$$\lim_{t\to\infty} e^{-\rho t}(\lambda_0 e^{\rho t})S(t) = 0 \Longrightarrow \lim_{t\to\infty} \lambda_0 S(t) = 0$$

This fundamental mathematical limit yields two distinct, mutually exclusive economic possibilities. The first possibility is pure physical exhaustion, where the total geological resource stock is entirely and completely depleted by the extraction sector ($S(\infty) = 0$) and the initial scarcity rent $\lambda_0$ remains strictly positive, driving prices to infinity. The second possibility is far more relevant and desirable to the macroeconomic circular economy transition. If the central planner successfully and continuously imposes the dynamically escalating Pigouvian tax trajectory $\tau(t) = \mu(t)$, the compounding environmental penalty levied on virgin extraction, combined symbiotically with exponentially decreasing marginal operational costs in rapidly maturing, heavily funded recycling technologies, will eventually force the first-order condition for virgin extraction into a permanent, hard corner solution ($q^* = 0$) well before the physical geological resource is completely depleted. Under this scenario, the unextracted resource stock successfully stabilizes permanently at a strictly positive boundary value $S(\infty) = S^* > 0$. We define this boundary case as economic exhaustion. In a perfectly circular steady state where primary virgin extraction ceases permanently and forever, virgin material is no longer the marginal source of consumption inputs. Consequently, the fundamental geological resource constraint entirely ceases to bind the macroeconomy. To mathematically satisfy the transversality limit when $S^* > 0$, the initial scarcity rent $\lambda_0$ must exactly and immediately equal zero. Consequently, $\lambda(t) = 0$ for all points in time t. Under these specific, idealized circular steady-state conditions, the classical Hotelling scarcity rent vanishes completely from the macroeconomic system. Long-term market pricing is governed no longer by geological scarcity, but entirely and exclusively by the environmental shadow cost of pollution μ and the operational logistics efficiency of the secondary recycling sector ($C_{r'}$).

Two features of this argument merit closer attention, since the transversality condition alone only rules out certain paths without establishing that a path with $q(t) = 0$ is itself optimal. First, the co-state equation $\dot{\lambda}(t) = \rho\lambda(t)$ derived above (Section 5.3) is linear, autonomous, and admits $\lambda(t) \equiv 0$ as its only non-explosive solution: any $\lambda_0 \neq 0$ implies $|\lambda(t)| \to \infty$, and because $\lambda(t) = \lambda_0 e^{\rho t}$ can equal zero at some date only if $\lambda_0 = 0$, economic exhaustion whenever $S^\infty > 0$ requires $\lambda(t) \equiv 0$ for all $t \geq 0$, not merely in the limit. The Hotelling scarcity rent is therefore absent from the entire program in this regime, not only from its long-run limit.

**Sufficiency of the boundary solution.** That $\lambda(t) \equiv 0$ is necessary for economic exhaustion does not by itself show that $q(t) = 0$ is optimal; this requires the Karush–Kuhn–Tucker multiplier $\eta_q(t)$ to satisfy dual feasibility, $\eta_q(t) \geq 0$. Evaluating the first-order condition of Section 5.1 at the corner with $\lambda(t)$ equal to zero gives

$$\eta_q(t) = \lambda(t) + \mu(t) + C_q'(0) - U'\big(r(t)\big)$$

so $q(t) = 0$ satisfies the necessary conditions if and only if

$$\mu(t) + C_q'(0) \geq U'\big(r(t)\big)$$

Because $U$ is concave and $C_q$, $C_r$, $D$ are convex by the assumptions of Section 4.1, the Hamiltonian is jointly concave in $q, r, W$ for any given co-state values each term is either concave (a concave function of a linear aggregate, or the negative of a convex function) or linear. The Mangasarian sufficiency theorem for the maximum principle then applies: any admissible path satisfying the first-order and complementary-slackness conditions, the co-state equations, and the transversality condition is optimal, not merely undominated. The boxed condition above is therefore both necessary and jointly with the interior conditions away from the corner sufficient for $q(t) = 0$ to belong to the optimal path. Whether it holds depends on the specific functional forms of $U$, $C_q$, and $D$ and on the prevailing values of $r$ and $\mu(t)$; it is not guaranteed for every parameterization of the model, which is why we state it as a condition to be verified rather than as a general property of the framework.

### *3.6. Decentralization: Market Implementation of the Planner's Solution*

The preceding results describe the allocation chosen by a benevolent social planner. This subsection shows how the same allocation can be implemented as a competitive equilibrium once government sets two Pigouvian instruments, making precise the informal claim in Section 5.2 that $\tau(t) = \mu(t)$ corrects only the environmental externality rather than the resource-scarcity rent.

Consider a representative extraction firm that owns the resource stock $S(t)$, sells output at the competitive price $p(t)$ of the consumption good, pays a per-unit extraction tax $\tau(t)$, and incurs the cost $C_q(q)$. The firm solves

$$\max_{q(t)\geq 0} \int_0^\infty e^{-\rho t}\left[\big(p(t) - \tau(t)\big)q(t) - C_q\big(q(t)\big)\right]dt \quad \text{s.t. } \dot{S}(t) = -q(t)$$

Its private current-value Hamiltonian is

$$\mathcal{H}^f = (p - \tau)q - C_q(q) - \lambda_f q$$

with first-order condition

$$p(t) = \tau(t) + C_q'\big(q(t)\big) + \lambda_f(t)$$

and a co-state equation for $\lambda_f(t)$ of the same form as the planner's, since extraction costs are independent of the remaining stock (Section 4.1): $\dot{\lambda}(t) = \rho\lambda(t)$. Because this is the same equation that governs the planner's $\lambda(t)$, under rational expectations the firm's privately optimal shadow value of the resource coincides with the planner's, $\lambda_f(t) = \lambda(t)$ competitive ownership of the resource stock already internalizes the Hotelling scarcity rent without any tax on it.

A representative recycling firm sells output at the same price $p(t)$, incurs cost $C_r(r)$, and pays a residue tax $\tau_r(t)$ on its output, reflecting the toxic byproduct it generates:

$$\max_{r(t)\geq 0}\big[p(t)r(t) - C_r\big(r(t)\big) - \tau_r(t)r(t)\big] \;\Rightarrow\; p(t) = C_r'\big(r(t)\big) + \tau_r(t)$$

A representative consumer with utility $U(C)$ takes $p(t)$ as given and sets the standard optimality condition

$$U'\big(C(t)\big) = p(t)$$

Setting the two tax instruments equal to the planner's shadow prices, $\tau_r(t) = \beta\mu(t)$, and $\tau(t) = \mu(t)$, reproduces exactly the planner's first-order conditions of Section 5.1 in the decentralized economy: the extraction sector's condition becomes $p = C_q'(q) + \lambda + \mu$, and the recycling sector's becomes $p = C_r'(r) + \beta\mu$, matching the social optimum term for term.

**Proposition 3 (Decentralization).** Under the maintained assumptions of Sections 4 and 5, the socially optimal allocation can be implemented as a competitive equilibrium by an upstream extraction tax $\tau(t) = \mu(t)$ together with a downstream recycling-residue tax $\tau_r(t) = \beta\mu(t)$, given competitive ownership of the resource stock $S$. The extraction tax alone, without well-defined property rights over $S$ or without the complementary residue tax on recycling, does not in general decentralize the social optimum, since firms would then have no private incentive to internalize either the scarcity rent or the toxic byproduct of recycling.

This result also gives the downstream "waste hierarchy tax" discussed in Section 8.2 a precise counterpart in the present framework: $\tau_r(t) = \beta\mu(t)$ is the schedule that a recycling-residue instrument should track if it is to implement, rather than merely complement, the upstream extraction tax.

### *3.7 Relaxing Baseline Assumptions: Imperfect Substitution and Boundary States*

The baseline optimal control model provides useful theoretical insights under its specific assumptions. A complete theoretical analysis should also explore how the dynamic system fundamentally behaves when core assumptions are relaxed or when structural parameters approach extreme mathematical boundary values.

#### *3.7.1. Constant Elasticity of Substitution (CES) Aggregation*

The primary baseline model robustly assumes that total aggregate consumption is simply the linear sum of virgin and recycled material flows ($C = q + r$), implying perfect substitutability between the two inputs. In practice, secondary recycled materials frequently suffer from quality degradation such as shortened polymer chains during plastic reheating and therefore may not serve as perfect engineering substitutes for virgin inputs across all required manufacturing tolerances. To evaluate the stability of the perfect circularity outcome, we also examine a more flexible CES form, which allows imperfect substitution between materials:

$$C_{CES}(t) = \left[\theta q(t)^{\frac{\sigma-1}{\sigma}} + (1-\theta)r(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Where $\sigma > 0$ dictates and governs the strict elasticity of substitution between virgin and recycled inputs, and the parameter $\theta \in (0,1)$ represents the distributional structural share favoring either primary or secondary inputs within the economy. If the substitution parameter $\sigma \to \infty$, the inputs behave as perfect substitutes, and the CES model reduces to the linear baseline.

However, if $\sigma < \infty$, the inputs are imperfect substitutes. Under the conditions of imperfect substitutability, the marginal utility derived specifically from secondary recycling depends heavily and continuously on the contemporaneous rate of primary virgin extraction. If the standard microeconomic Inada

conditions hold for the defined CES function, the marginal utility of virgin extraction approaches infinity as the quantity of virgin extraction is pushed toward zero by the planner:

$$\lim_{q\to 0}\frac{\partial U}{\partial q}=\infty$$

Because the marginal physical cost of extraction ($C_{q'}$) and the derived environmental penalty (μ) are finite, the fundamental first-order condition for virgin extraction cannot reach the boundary state where $q=0$. The macroeconomy will continuously demand a non-zero input of virgin material to physically blend with secondary recycled stock to maintain structural engineering integrity. Therefore, under conditions of imperfect substitutability, a fully circular steady state cannot be reached asymptotically; the macroeconomy can only approach a state of optimized "near-circularity," and the Hotelling scarcity rent (λ) will never completely vanish from the system dynamics.

To make this claim precise, the optimal control problem of Section 5 can be rewritten with $q$+$r$ replaced by the CES aggregator $C_{CES}(q,r)$ throughout. The current-value Hamiltonian becomes

$$\mathcal{H}_{CES}=U\big(C_{CES}(q,r)\big)-C_q(q)-C_r(r)-D(W)-\lambda q \\ -\mu(q+\beta r-\alpha W)+\eta_q q+\eta_r r$$

with $\eta_q$, $\eta_r$ the same complementary-slackness multipliers as before. Differentiating $C_{CES}(q,r)$ gives the CES marginal products

$$\frac{\partial C_{CES}}{\partial q}=\theta\left(\frac{C_{CES}}{q}\right)^{1/\sigma},\qquad \frac{\partial C_{CES}}{\partial r}=(1-\theta)\left(\frac{C_{CES}}{r}\right)^{1/\sigma}$$

so the first-order conditions for an interior solution are

$$U'(C_{CES})\,\theta\left(\frac{C_{CES}}{q}\right)^{1/\sigma}-C_q'(q)-\lambda-\mu+\eta_q=0$$

$$U'(C_{CES})\,(1-\theta)\left(\frac{C_{CES}}{r}\right)^{1/\sigma}-C_r'(r)-\beta\mu+\eta_r=0$$

and dividing one by the other yields the CES tangency condition that replaces the linear arbitrage equation of Section 5.1:

$$\frac{\theta}{1-\theta}\left(\frac{r}{q}\right)^{1/\sigma}=\frac{C_q'(q)+\lambda+\mu}{C_r'(r)+\beta\mu}$$

which reduces to the baseline condition $C_r'(r)=C_q'(q)+\lambda+\mu$ in the limit $\sigma\to\infty$, since

$$\lim_{\sigma\to\infty}C_{CES}(q,r)=q+r$$

The Karush–Kuhn–Tucker condition for the corner $q=0$ now fails for a different, more fundamental reason than in the baseline model. As $q$ approaches zero from above with $\theta\in(0,1)$ and $\sigma<\infty$, the CES marginal product $\partial C_{CES}/\,\partial q$ diverges,

$$\lim_{q\to 0^+}\theta\left(\frac{C_{CES}}{q}\right)^{1/\sigma}U'(C_{CES})=\infty\qquad\big(\sigma<\infty,\ \theta\in(0,1)\big)$$

so the marginal-utility term in the first-order condition for $q$ grows without bound while $C_q'(0)$, $\lambda(t)$, and $\mu(t)$ remain finite. No finite $\eta_q\geq 0$ can then satisfy the stationarity condition at $q=0$, so the corner is never a Karush–Kuhn–Tucker point: an interior $q^*>0$ is required at every finite date. This is the Inada-type property referenced informally above, now derived directly from the first-order conditions rather than asserted from the shape of the production function alone and it parallels the corresponding Inada condition that drives the analogous non-corner result in Bertarelli et al. [4] (Section 4), who obtain the same qualitative feature in a related finite-horizon, labor-allocation CES model of recycled and non-recycled resource use. It also shows that the qualification is not a matter of degree: for any $\sigma<\infty$ the economy cannot converge to $q=0$, however close $\sigma$ is to infinity; only in the non-generic limiting case $\sigma\to\infty$ (perfect substitutability, the baseline model of Sections 4–5) does the corner solution of Section 5.4 become attainable. The comparative-statics question of how the steady-state extraction rate $q^*(\sigma)$ varies with $\sigma$ over the admissible range is left for numerical analysis in future work, since closed-form solutions are not generally available once both $C_q$ and $C_r$ are strictly convex.

### *3.7.2. Boundary Analysis: Absorption Capacities and Technological Frontiers*

The Zero-Absorption Scenario (The Plastics Crisis): If the global ecosystem possesses no biological or chemical capacity to naturally digest the generated synthetic pollutants, the natural absorption parameter approaches zero ($\alpha\to 0$). This boundary modification models persistent synthetic polymers, perfluoroalkyl substances (PFAS), and forever chemicals [2]. Under this extreme boundary condition, the effective mathematical discount rate of long-term environmental damages drops from $(\rho+\alpha)$ to simply ρ. The definite integral defining the shadow cost becomes heavily and irreversibly weighted toward the infinite future. Unless the recycling process is perfectly clean, the optimal dynamic Pigouvian tax rises sharply, leading total societal consumption to contract severely in the long run to prevent infinite, apocalyptic damage accumulation.

The Perfectly Clean Recycling Utopia: If technological breakthroughs achieve near-perfect recycling efficiency in the secondary processing sector, the toxicity parameter drops to zero ($\beta=0$). The complex inter-sectoral arbitrage equation simplifies to $C_{r'}(r)=C_{q'}(q)+\lambda+\mu$. Because recycling generates no toxic byproduct, the central planner immediately ceases to penalize the secondary recycling flow entirely. The macroeconomy transitions to a stable, circular state constrained exclusively and solely by the operational logistics costs of recycling ($C_{r'}$).

The Absence of Recycling Technology: In a purely linear historical economy where recycling technology is entirely unavailable or deemed prohibitively expensive across all sectors ($r=0$), the two-variable control problem reduces to a standard exhaustible resource depletion model permanently paired with cumulative pollution. The environmental shadow cost μ grows continuously with unabated extraction, rapidly accelerating the arrival of economic exhaustion. Society would then need to move quickly to alternative backstop technologies outside the primary material paradigm to survive.

### *3.8. Geometric Stability and Comparative Statics*

#### *3.8.1. State-Costate Phase Space Analysis*

To evaluate whether the proposed fully circular state represents a stable steady-state equilibrium rather than an explosive trajectory, the Jacobian matrix of the linearized continuous dynamical system must be analyzed at the steady state. At the specific boundary equilibrium where all virgin extraction ceases, the macroeconomic system reduces to a two-dimensional state-costate phase space comprising strictly the variables $(W,\mu)$. The differential equations describing this phase space are:

$$\dot{W}=\beta r-\alpha W$$

$$\dot{\mu}=(\rho+\alpha)\mu-D'(W)$$

By mathematically setting $\dot{W}=0$ and $\dot{\mu}=0$,the steady-state coordinates$(W^*,\mu^*)$ are established.

At this specific mathematical boundary, the recycling rate r is determined continuously by the modified first-order condition $U'(r)-C_{r'}(r)=\beta\mu$. Because both marginal utility and marginal cost are strictly defined functions of r, the secondary recycling rate is an implicit function of the environmental shadow cost μ. Differentiating this implicit function with respect to μ yields:

$$\frac{\partial r}{\partial\mu}=\frac{\beta}{U''(r)-C_{r''}(r)}$$

Since $U''<0$ and $C_{r''}>0$ strictly due to the baseline convexity assumptions established in Section 4, the denominator is negative, so the derivative $\partial r/\,\partial\mu<0$. Economically, this directly implies that as the shadow cost of pollution increases in the economy, the required recycling effort must dynamically and automatically decrease to actively mitigate the compounding generation of its own toxic sludge byproduct ($\beta r$).

The Jacobian matrix J of the continuous dynamical system evaluated precisely at this specific steady state is structurally constructed as:

$$J=\begin{bmatrix}\frac{\partial\dot{W}}{\partial W} & \frac{\partial\dot{W}}{\partial\mu}\\ \frac{\partial\dot{\mu}}{\partial W} & \frac{\partial\dot{\mu}}{\partial\mu}\end{bmatrix}=\begin{bmatrix}-\alpha & \beta\frac{\partial r}{\partial\mu}\\ -D''(W) & \rho+\alpha\end{bmatrix}$$

To determine the topological geometric stability of the equilibrium within the dynamic system, the trace and determinant of the Jacobian matrix are computed:

$$\text{Trace}(J)=-\alpha+(\rho+\alpha)=\rho$$

$$\text{Determinant}(J)=-\alpha(\rho+\alpha)-\beta\frac{\partial r}{\partial\mu}(-D''(W)) \\ =-\alpha(\rho+\alpha)+\beta D''(W)\frac{\partial r}{\partial\mu}$$

Because the intergenerational social discount rate strictly guarantees $\rho>0$, the mathematical trace is strictly positive. Given that biological absorption $\alpha>0$,

damage convexity $D'' > 0$, and the derived implicit function $\partial r / \partial \mu < 0$, both discrete terms comprising the determinant calculation are strictly negative. This specific matrix configuration guarantees that $\text{Determinant}(J) < 0$. In a strictly two-dimensional state-costate Hamiltonian continuous system, a negative determinant mathematically dictates with certainty that the eigenvalues of the Jacobian matrix are real and of exactly opposite signs. This provides the rigorous, definitive proof that the circular economy transition state represents a saddle-path stable equilibrium. Consequently, there exists exactly one unique, convergent stable trajectory along which the central planner can seamlessly guide the economy toward the steady state via the precise, continuous application of the dynamic Pigouvian tax $\tau(t) = \mu(t)$.This local analysis is confined to the two-dimensional $(W, \mu)$ subsystem, and it is worth verifying explicitly that this restriction is exact rather than an approximation. In the economic-exhaustion regime, $q(t) = 0$ and, by the argument of Section 5.4, $\lambda(t) \equiv 0$ throughout; consequently $\dot{S} \equiv 0$ and $\dot{\lambda} \equiv 0$ identically, so $S$ and $\lambda$ are constants of the motion rather than dynamic variables once the economy is in this regime. Because neither $\dot{W}(t)$ nor $\dot{\mu}(t)$ depends on $S$ or $\lambda$ both are functions of $(W, \mu)$ alone, as the equations above show the Jacobian of the full four-dimensional system $(S, \lambda, W, \mu)$, evaluated at the boundary steady state, is block-diagonal: a 1×1 block with eigenvalue zero for $S$, a 1×1 block with eigenvalue $\rho$ (positive) for $\lambda$, and the 2×2 block for $(W, \mu)$ analyzed above. The zero eigenvalue in the $S$-direction reflects that the boundary value $S^*$ is pinned down by the extraction history prior to reaching the corner rather than by the local dynamics at it; it is not a failure of stability but a direct consequence of $q = 0$ at the boundary. The positive eigenvalue $\rho$ in the $\lambda$-direction confirms, independently of the transversality argument in Section 5.4, that any value $\lambda \neq 0$ is locally unstable and cannot be sustained. The genuine transitional dynamics of the post-exhaustion economy are therefore fully and exactly captured not merely approximated by the reduced $(W, \mu)$ system, and it is this reduced system, rather than the full four-dimensional one, for which saddle-path stability is established above.

#### *3.8.2. Proposition 2: The Intergenerational Social Discount Rate Paradox*

Comparative statics theorems permit the vital analytical evaluation of how the long-term circular equilibrium structurally shifts in direct, immediate response to exogenous perturbations in core structural parameters.

**Proposition 2.** Assuming strict concavity of utility, strict convexity of recycling costs, and a strictly positive recycling toxicity parameter ($\beta > 0$), an increase in the intergenerational social discount rate ($\rho$) mathematically forces a structural increase in the socially optimal equilibrium recycling effort ($r^*$).

Proof. The mathematical sensitivity and responsiveness of the equilibrium recycling rate ($r^*$) with respect to structural changes in the social discount rate ($\rho$) requires the careful application of the implicit function theorem. At the steady-state boundary where $q = 0$ and $\lambda = 0$, the system successfully reaches an equilibrium defined completely by $\dot{\mu} = 0$, which yields the exact shadow cost governing equation:

$$\mu = \frac{D'(W^*)}{\rho + \alpha}$$

Substituting this specific shadow cost back directly into the modified boundary first-order condition ($U'(r) - C_{r'}(r) = \beta\mu$) yields the core comparative static equation utilized for the proof:

$$U'(r) - C_{r'}(r) = \frac{\beta D'(W^*)}{\rho + \alpha}$$

To successfully isolate the direct effect of $\rho$ on $r^*$, implicit derivation logic is systematically applied. If the intergenerational social discount rate ($\rho$) strictly increases, the mathematical denominator on the right side of the equivalence equation expands. This mathematically forces the entire right side of the equation which physically represents the strictly discounted marginal environmental penalty of the recycling sector's own toxic residue to immediately decrease. Because $U'' < 0$ and $C_{r''} > 0$, the left side of the equation ($U'(r) - C_{r'}(r)$) acts as a strictly decreasing function of r. For the left side of the equation to actively decrease and maintain continuous algebraic equivalence with the newly lowered environmental penalty on the right side, the equilibrium recycling effort $r^*$ must structurally and physically increase. Therefore, the implicit derivation implies that $\partial r^* / \partial \rho > 0$. This implies a counter-intuitive policy mechanism. Standard environmental economics broadly and colloquially posits that higher societal discount rates are fundamentally detrimental to ecosystem conservation. However, in this specific, mathematically rigorous dynamic framework, an increase in the social discount rate signifies that the central planner places a very high, immediate time preference on present mass consumption, heavily discounting the fundamental welfare and environmental safety of future generations. Because the secondary recycling process is not perfectly clean ($\beta > 0$), it acts effectively as a delayed polluter. A high discount rate sharply reduces the calculated present value of these future environmental damages, directly forcing a large decline in the shadow cost of toxic pollution ($\mu$). Without a robust, high-value environmental shadow price to penalize the inherent thermodynamic inefficiency of secondary recycling, the central planner legally permits the polluting recycling sector to expand rapidly to satisfy present consumption demands. Conversely, a society with a low discount rate will heavily penalize the toxic sludge generated by the recycling process itself, forcing a significant reduction in overall physical recycling output $r^*$ to prioritize long-term preservation of the ecosystem over immediate consumption gratification.

Proposition 2 is a property of the baseline model rather than a general claim about discounting and conservation. The sign of $\partial r^* / \partial \rho$ depends on the recycling process itself being a source of environmental damage ($\beta$ positive) and on the specific functional forms of $U$, $C_r$, and $D$ assumed in Section 4.1; a higher discount rate lowers the shadow cost attached specifically to recycling's own toxic residue, and there is no comparably direct channel through which it would raise $r^*$ in a model without this feature. The result should therefore not be read as a general argument that higher social discount rates favor conservation-adjacent outcomes; it identifies one mechanism, operative under the maintained assumptions here, that runs opposite to the standard discounting intuition. Confirming its quantitative importance as distinct from its sign would require numerical sensitivity analysis across plausible ranges of $\beta$ and the curvature of $U$, $C_r$, and $D$, which is beyond the analytical scope of this paper and is left for future calibration work.

### *3.9. Policy Context and External Empirical Evidence*

The continuous-time optimal control framework developed in Sections 4–7 characterizes the tax trajectory required for an efficient transition and derives the market instruments that implement it (Section 5.5), but it is not calibrated or numerically simulated in this paper: no data have been fitted to the functions $U$, $C_q$, $C_r$, or $D$, and no numerical path for $\tau(t)$ has been computed. The results are therefore theoretical and are not empirically validated here. What follows relates the qualitative direction of the theoretical results a shift of fiscal burden from labor toward virgin material extraction to existing external evidence on comparable tax-shifting reforms, and discusses complementary downstream policy design; this evidence is contextual, drawn from a different model calibrated to a different economy, and should not be read as a quantitative test of the framework above.

#### *3.9.1. Tax Revenue Recycling: External Evidence and Its Limits*

A dynamic upstream extraction tax of the kind derived in Section 5 raises a standard political-economy objection: taxing virgin material extraction could contract output and employment in the affected sectors. Because the present model has no labor market, capital stock, wage determination, or any endogenous channel through which tax revenue re-enters the economy, this objection cannot be evaluated within the model itself. The double-dividend hypothesis in environmental economics holds that a contraction need not follow if tax revenue is recycled back into the economy for instance, by lowering distortionary labor taxes. To illustrate the plausible order of magnitude of such an effect, we report results from an existing external computable general equilibrium (CGE) study rather than from the model developed here: Cambridge Econometrics' simulation for The Ex'tax Project [9], covering a structural shift of taxation from labor to natural resources across the 27 European Union member states then in the bloc. Over a simulated five-year horizon (introduced linearly from 2016 and scaled to full effect by 2020), that scenario modeled a cumulative €1,716 billion transfer of tax revenue from labor to natural resources, virgin extraction, and primary consumption [9], with the macroeconomic effects summarized in Table 2.

**Table 2.** Illustrative empirical impacts of a €1,716 billion structural tax shift from labor to natural resources across 27 EU nations.

| Macroeconomic Indicator | Modeled Systemic Impact (5-Year Horizon) | Strategic Policy Implication and Justification |
|---|---|---|
| Gross Domestic Product (GDP) | +2.0% (€842 billion added regional growth) | Suggests environmental tax implementation helps avoid structural economic recession if revenue is fully recycled. |
| Net Employment Creation | +6.6 million new jobs generated | Highlights the high labor-intensity of localized circular recycling, complex municipal sorting, and industrial product repair. |
| Greenhouse Gas Emissions | -8.2% (1,038 million tonnes saved) | Secures rigorous national compliance with critical limits on atmospheric toxic accumulation and global climate treaty targets. |
| Energy Resource Use | -194 million tonnes oil equivalent | Directly and effectively curtails the linear extraction of primary non-renewable natural resources across 12 distinct energy types. |
| Fossil Fuel Imports | €27.7 billion external bill reduction | Improves regional supply-chain security by reducing exposure to volatile global commodity markets. |

Source: Ref [10]

Note: these figures describe a separate model calibrated to a different economy; they are reported as external policy context, not as a calibration or simulation of the theoretical framework in Sections 4–7.

These external figures are consistent with the qualitative direction of the theoretical result that shifting fiscal burden from labor to virgin material extraction need not imply macroeconomic contraction but they do not test the specific tax trajectory $\tau(t) = \mu(t)$ derived in Section 5, which has not itself been simulated. A direct empirical evaluation of the present framework would require calibrating $U$, $C_q$, $C_r$, and $D$ to data for a specific economy and numerically solving the resulting optimal control problem; this is a natural extension for future work but falls outside the scope of the present theoretical contribution.

#### *3.9.2. Downstream Policy Complements and the Waste Hierarchy Tax*

The theoretical derivation shows that, under the model's assumptions, $\tau(t) = \mu(t)$ is the optimal upstream fiscal policy. Imposing a tax solely at the point of upstream extraction, however, transmits a diluted price signal to downstream product designers, municipal planners, and household consumers. To achieve the mathematical optimum in decentralized real-world markets, the upstream tax needs to be paired with downstream complementary policies a requirement Proposition 3 (Section 5.5) gives a precise theoretical counterpart to: implementing the social optimum requires not only the upstream tax $\tau(t) = \mu(t)$ but also the downstream recycling-residue tax $\tau_r(t) = \beta\mu(t)$, without which decentralized recycling firms have no incentive to internalize the toxic byproduct of their own activity.

Drawing on Milios' circular economy taxation framework, the model's logic is naturally complemented by a downstream "waste hierarchy tax" applied at the point of final waste treatment [3]. Under this framework, policy targets three stages of the product production, use, and final waste management and at the final stage, different treatment options are taxed in proportion to their environmental impact.

In Sweden, for example, landfilling is taxed comparatively heavily (around 500 SEK per tonne), incineration is taxed at an intermediate rate (around 300 SEK), and material recycling is taxed lightly or subsidized. This stratification prices the toxic-residue parameter $\beta$ the same parameter that enters the continuous-time model above into the municipal waste system directly. Comparable frameworks, such as the Advance Disposal Fee used in the European Union and the simpler Disposal Fee structures used in some developing economies including India [10], illustrate the range of downstream instruments available. Without such downstream complements, an upstream virgin-material tax acts as a second-best instrument on its own, since decentralized recyclers otherwise lack a direct incentive to invest in costly sorting and processing; layered this way, these taxes are generally considered growth-friendly relative to labor taxation, since they price an externality rather than a productive input [8, 3, 1].

## 4. Conclusions

This study develops a continuous-time optimal control model to characterize the macroeconomic transition from linear extraction to a circular economy. By distinguishing between total material flows and the accumulation of unrecoverable toxic pollutants, the framework captures both virgin resource depletion and the generation of persistent waste residues from secondary recycling dynamics constrained by thermodynamic limits rather than simple mass balance. The model constructs an augmented current-value Hamiltonian and applies Karush-Kuhn-Tucker conditions to derive the optimal Pigouvian tax on virgin extraction as the discounted present value of future marginal environmental damages. A central theoretical result establishes that at economic exhaustion when the economy reaches a perfectly circular steady state and virgin stocks remain in situ the classical Hotelling rule no longer applies: the initial shadow price of the virgin resource collapses to zero, as scarcity rents are superseded by environmental externality considerations. This allocation is decentralizable: Section 5.5 shows that a competitive equilibrium reproduces the social optimum once virgin extraction is taxed at $\tau(t) = \mu(t)$ and recycling's own toxic residue is taxed at $\tau_r(t) = \beta\mu(t)$, given competitive ownership of the resource stock.

The model suggests that higher discount rates lower the pollution shadow cost and may increase recycling intensity despite the environmental damage from recycling itself. This counter-intuitive result results from underweighting future environmental costs by present-biased planners.

The baseline model assumes perfect substitutability between virgin and recycled inputs, abstracting from material degradation, contamination, and structural fatigue in secondary feedstocks. Under finite elasticity of substitution, complete circularity is asymptotically unattainable, and residual virgin extraction remains necessary to avoid infinite marginal utility. The aggregation of heterogeneous pollutants into a single waste variable further simplifies biological decay rates and spatial dispersion. Future extensions incorporating multi-pollutant vector dynamics and spatially explicit damage functions would strengthen the model's empirical applicability to global supply chains.

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### Author Contributions

All authors contributed equally to the conception, preparation, and writing of this manuscript.

### Conflict of Interest

The authors, while observing publication ethics in the referencing, declare the absence of interest of conflict.

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